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Markov Interlacing Property for Perfect Splines

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We prove that if the zeros of two perfect splines p and q interlace, then the zeros of p' and q' also interlace. This is an extension of the classical result concerning algebraic polynomials proved by V. A. Markov.

Key Words: interlacing property; perfect splines; V. A. Markov; Tchebycheff systems.

1. INTRODUCTION

In his fundamental paper [10] Vladimir Markov studied the problem of estimating the value of the kth derivative of a polynomial P of degree n, $0<k\leqslant n$, in terms of its uniform norm. He established the remarkable inequality

$$\|P^{(k)}\| \leqslant \|T_n^{(k)}\| \, \|P\|, \qquad k=1, ..., n.$$

Here $\|\cdot\|$ denotes the uniform norm on $[-1, 1]$ and $T_n(x)$ is the nth degree Tchebycheff polynomial. The results and the methods developed in this paper have been so important that even more than 20 years after its appearance, a German translation of a slightly shorter version was republished in Mathematische Annalen with a preface by Professor S. N. Bernstein. A major ingredient in Markov's ingenious proof is the following lemma concerning the zeros of the algebraic polynomials.

Lemma of Markov. *Assume that the zeros of the polynomials $u(x):=(x-x_1)\cdots(x-x_n)$ and $v(x):=(x-y_1)\cdots(x-y_n)$ satisfy the interlacing conditions*

$$x_1 \leqslant y_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant y_n.$$

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Then the zeros $t_1 \leqslant \cdots \leqslant t_{n-1}$ *of* $u'(x)$ *and the zeros* $\tau_1 \leqslant \cdots \leqslant \tau_{n-1}$ *of* $v'(x)$ *interlace too, that is,*

$$t_1 \leqslant \tau_1 \leqslant t_2 \leqslant \cdots \leqslant t_{n-1} \leqslant \tau_{n-1}. \tag{1}$$

Moreover, if $x_1 < \cdots < x_n$ *and at least one of the inequalities* $x_i \leqslant y_i$, $i = 1, ..., n$, *is strict, then all inequalities in* (1) *are strict.*

Letting y_n tend to infinity one may obtain a similar result for polynomials u and v of degree n and $n-1$, respectively.

Markov's lemma became a delicate and important tool in the study of various extremal problems for algebraic polynomials (see [17]). In particular it has been very useful in the works concerning the estimation of functionals of P' in the space of polynomials (see [1, 4, 5, 13–15, 17]).

We are going to prove a similar result for perfect splines. Recall that a *perfect spline* of degree r with knots $\xi_1 < \cdots < \xi_{n-r}$ is any expression of the form

$$p(t) = \sum_{j=1}^{r} \alpha_j t^{j-1} + c \left[t^r + 2 \sum_{i=1}^{n-r} (-1)^i (t - \xi_i)_+^r \right]$$

with real coefficients $\{\alpha_j\}$ and c where, as usual, $t_+^m = t^m$ if $t \geqslant 0$, and zero otherwise. A characteristic property of the perfect spline $p(t)$ is that $|p^{(r)}(t)| = \text{const.}$ for each t except the knots $\xi_1, ..., \xi_{n-r}$. Perfect splines appear prominently in various extremal problems in classes of differentiable functions, and particularly in $W_\infty^r[a, b]$,

$$W_\infty^r[a, b] := \{ f \in C^{(r-1)}[a, b] : f^{(r-1)} \text{ abs. cont.}, \|f^{(r)}\|_\infty < \infty \},$$

$$\|g\|_\infty := \operatorname*{ess\,sup}_{t \in [a, b]} |g(t)|.$$

Just to give some examples recall that Karlin [6, 7] (see also de Boor [3]) proved the following: Given any data $\{x_i, f_i\}_{i=1}^{n+1}$, $\alpha \leqslant x_1 < \cdots < x_{n+1} \leqslant b$, there exists a perfect spline s of degree r with at most $n-r$ knots such that

$$s(x_i) = f_i \qquad \text{for} \quad i = 1, ..., n+1.$$

Moreover, s is the smoothest function in $W_\infty^r[a, b]$ that interpolates the data, that is,

$$\|s^{(r)}\|_\infty \leqslant \|f^{(r)}\|_\infty$$

for each f from $W_\infty^r[a, b]$ satisfying the interpolation conditions $f(x_i) = f_i$, $i = 1, ..., n+1$. An immediate consequence thereof is the following result which is known as the *fundamental theorem of algebra* for perfect splines.

THEOREM A. *Given the points* $\mathbf{x} = (x_1, ..., x_n)$, $x_1 < \cdots < x_n$, *there exists a unique* (*up to multiplication by a constant*) *non-trivial perfect spline of degree p with at most* $n - r$ *knots satisfying the conditions*

$$p(x_i) = 0, \qquad i = 1, ..., n.$$

Moreover, p has exactly $n - r$ *knots.*

The proof of this theorem in a more general form involving multiple zeros (and Birkhoff type zeros), as well as other properties of perfect splines can be found, for example, in [2].

The perfect spline p of Theorem A, normalized by the condition $p^{(r)}(x_n) = 1$, will be denoted in this paper by $p(\mathbf{x}; t)$.

The next important extremal property of $p(\mathbf{x}; t)$ (proved in [11]), which in and of itself would justify the interest in these functions, is the following: Let $a \leqslant x_1 < \cdots < x_n \leqslant b$. Then for every $t \in [a, b]$,

$$\max\{|f(t)|: f \in W^r_\infty[a, b], \|f^{(r)}\|_\infty \leqslant 1, f(x_i) = 0, i = 1, ..., n\} = |p(\mathbf{x}; t)|.$$

Therefore the perfect spline vanishing at $\mathbf{x}$ represents the error of the best method of approximation of functions from $W^r_\infty[a, b]$ on the basis of their values at $\mathbf{x}$ (see [2] for details).

Perfect splines can be considered as natural generalizations of algebraic polynomials (which are perfect splines without knots). Thus one would expect that some of the properties of the algebraic polynomials are inherited by their generalizations. In particular, both simple examples and computer simulations have indicated that Markov's interlacing property seems to hold for perfect splines. Here we give proof of this fact.

This result could be useful in the study of Kolmogorov type inequalities concerning estimation of the derivatives of functions from $W^r_\infty[a, b]$ which are bounded on $[a, b]$ or on certain discrete subsets. An interesting problem of this kind was solved by Pinkus [12].

2. MARKOV'S LEMMA FOR GENERALIZED POLYNOMIALS

Our approach is based on the following observation mentioned already in [1].

Markov's interlacing property is equivalent to the statement: *Each zero* η *of the derivative of an algebraic polynomial* $P(x) := (x - x_1) \cdots (x - x_n)$ *is a strictly increasing function of* x_k *in the domain* $x_1 < \cdots < x_n$.

We shall prove the equivalence even for generalized polynomials with respect to an arbitrary Tchebycheff system. Note that having shown this equivalence the Markov lemma would follow easily since, as is known, every zero η of the derivative of P is an increasing function of x_k. A short proof of this is obtained by differentiating the identity

$$0=\frac{P'(\eta)}{P(\eta)}=\sum_{i-1}^{n}\frac{1}{\eta-x_i}$$

with respect to x_k. One gets

$$0=-\left(\sum_{i=1}^{n}\frac{1}{(\eta-x_i)^2}\right)\frac{\partial\eta}{\partial x_k}+\frac{1}{(\eta-x_k)^2}.$$

This yields

$$\frac{\partial\eta}{\partial x_k}>0$$

and thus, the strict monotonicity of η.

The above observation provides a new proof of Markov's lemma which admits extensions to other systems of polynomial like functions. The main steps in this proof are: (i) the equivalence and (ii) the monotonicity. We shall illustrate our method by fulfilling both of these two tasks on a certain class of smooth generalized polynomials. Then, in the next section, we shall apply this approach to perfect splines with even more care.

We first describe the class of generalized polynomials.

Let $\bar{\varphi}:=\{\varphi_1(x), ..., \varphi_n(x)\}$ be an arbitrary system of continuous functions on $[a, b]$. Assume that $\bar{\varphi}$ is a Tchebycheff system (or briefly, T-system) on $[a, b]$, that is,

$$\det\begin{bmatrix}\varphi_1(t_1) & \cdots & \varphi_n(t_1)\\ \vdots & \ddots & \vdots\\ \varphi_1(t_n) & \cdots & \varphi_n(t_n)\end{bmatrix}\neq 0$$

for each $\mathbf{t}=(t_1, ..., t_n)$ such that $a\leqslant t_1<\cdots<t_n\leqslant b$. Assume, in addition, that the functions $\{\varphi_1(x), ..., \varphi_{n-1}(x)\}$ also constitute a T-system on $[a, b]$. Consider the induced system $\bar{\phi}:=\{\phi_0, ..., \phi_n\}$ defined by

$$\phi_0(x)=1,$$

$$\phi_k(x)=\int_a^x \varphi_k(t)\,dt, \qquad k=1, ..., n.$$

It is a Tchebycheff system too. Besides, $\{\phi_0, ..., \phi_{n-1}\}$ is a T-system on $[a, b]$. Then fore every given set of points $\mathbf{x}=(x_1, ..., x_n)$ with $a\leqslant$

$x_1 < \cdots < x_n \leqslant b$, there exists a unique generalized polynomial ϕ of the form

$$\phi(x) = \phi_n(x) + \sum_{k=0}^{n-1} \alpha_{k+1} \phi_k(x)$$

which vanishes at $x_1, ..., x_n$. We shall denote this polynomial by $\phi(\mathbf{x}; t)$. Note that $\phi(\mathbf{x}; t)$ has no other zeros except $\mathbf{x}$ and $\phi'(\mathbf{x}; x_i) \neq 0$ for $i = 1, ..., n$. Indeed, otherwise Rolle's theorem would imply that $\phi'(\mathbf{x}; t)$ has at least n distinct zeros, a contradiction to the assumption that $\varphi_1, ..., \varphi_n$ is a Tchebycheff system. Thus $\phi(\mathbf{x}; t)$ changes sign in $x_1, ..., x_n$ and clearly each subinterval (x_i, x_{i+1}) contains exactly one zero of $\phi'(\mathbf{x}; t)$.

The next property of the generalized polynomials $\phi(\mathbf{x}; t)$ will play a central role in the sequel.

Lemma 1. *Assume that the points* $\mathbf{x} = (x_1, ..., x_n)$, $x_1 < \cdots < x_n$, *and* $\mathbf{y} = (y_1, ..., y_n)$ *interlace, that is,*

$$x_1 \leqslant y_1 \leqslant \cdots \leqslant x_n \leqslant y_n,$$

with at least one strict inequality $x_i < y_i$. *Then there is no point* $\eta \in [a, b]$ *for which*

$$\phi'(\mathbf{x}; \eta) = \phi'(\mathbf{y}; \eta) = 0.$$

Proof. Assume the contrary. Let η be a point from $[a, b]$ for which $\phi'(\mathbf{x}; \eta) = \phi'(\mathbf{y}; \eta) = 0$. Since η is a critical point of $\phi(\mathbf{x}; t)$, it should lie between two consecutive zeros of $\phi(\mathbf{x}; t)$, say in (x_j, x_{j+1}). Note that $\phi(\mathbf{y}; \eta) \neq 0$. Indeed, if $\phi(\mathbf{y}; \eta) = 0$, then $\eta = y_j$ and y_j would be a double zero of $\phi(\mathbf{y}; t)$, a contradiction to the remark above. Consider the polynomial

$$Q(t) := \phi(\mathbf{x}; t) - \alpha \phi(\mathbf{y}; t) \qquad \text{with} \quad \alpha = \frac{\phi(\mathbf{x}; \eta)}{\phi(\mathbf{y}; \eta)}.$$

Clearly $Q(\eta) = Q'(\eta) = 0$. If $\mathbf{x}$ and $\mathbf{y}$ interlace strictly (i.e., if $x_1 < y_1 < x_2 < \cdots < x_n < y_n$), then

$$Q(x_i)\, Q(x_{i+1}) = \alpha^2 \phi(\mathbf{y}; x_i)\, \phi(\mathbf{y}; x_{i+1}) < 0$$

and $Q(t)$ has an odd number of zeros in each subinterval (x_i, x_{i+1}), $i = 1, ..., n-1$. Thus Q has at least one zero in (x_i, x_{i+1}) for $i \neq j$. Beside, $Q(t)$ has a double zero at η and changes sign in a certain point $t_0 \in (x_j, x_{j+1})$. There are two possibilities:

(i) $t_0 \neq \eta$. In this case Q has at least $n+1$ zeros situated at n distinct points (η is a double zero). Then, by Rolle's theorem, Q' would have at least n distinct zeros. Since $Q \in \operatorname{span}\{\varphi_1, ..., \varphi_n\}$ and $\bar{\varphi}$ is a T-system, this implies $Q' \equiv 0$ and consequently $Q \equiv 0$, which is impossible since the zeros of $\phi(\mathbf{x}; t)$ and $\phi(\mathbf{y}; t)$ do not coincide.

(ii) $t_0 = \eta$, i.e., $Q(t)$ has a double zero at η and changes sign there. Then $Q'(t)$ vanishes at η and does not change sign in a neighborhood of η. Hence η can be counted as a double zero for Q'. In addition, Q' has at least $n-2$ other zeros out of (x_j, x_{j+1}). But $Q' \in \operatorname{span}\{\varphi_1, ..., \varphi_n\}$ and $\bar{\varphi}$ is a T-system. Then, by a known result about continuous T-system (see [9, Theorem 1.1, Chap. II], the total number of zeros of Q', counting the double zeros twice, should not exceed $n-1$, provided Q' is a non-zero polynomial. So, we conclude that $Q \equiv 0$.

In the remaining case when not all inequalities $x_i \leqslant y_i$ re strict one can see similarly that Q' has at least n zeros, counting eventually η as a double zero. As a hint, consider for example the case when

$$x_i = y_i \quad \text{for } i = k, ..., k+m-1, \qquad \text{and} \qquad x_i < y_i, \quad i = k-1, i = k+m.$$

Clearly $\phi(\mathbf{y}; t)$ has exactly $m+1$ zeros in (x_{k-1}, x_{k+m}). Assume that m is odd. Then $\phi(\mathbf{y}; t)$ has an even number of sign changes in (x_{k-1}, x_{k+m}) and therefore

$$\operatorname{sign} Q(x_{k-1}) = -\operatorname{sign} \alpha\phi(\mathbf{y}; x_{k-1}) = -\operatorname{sign} \alpha\phi(\mathbf{y}; x_{k+m}) = \operatorname{sign} Q(x_{k+m}).$$

Then Q has also an even number of zeros in (x_{k-1}, x_{k+m}). This shows that in addition to $x_k, ..., x_{k+m-1}$, Q has at least one more zero in (x_{k-1}, x_{k+m}). Thus, to each subinterval (x_i, x_{i+1}), $i = k-1, ..., k+m-1$, we can assign a zero of Q. Note that η, begin a double zero of Q, is not counted here. In this way we come again to the conclusion that Q has in total at least $n+1$ zeros in $[a, b]$. This leads to a contradiction like in the previous case. The proof is complete. ∎

The Equivalence. We shall show that the Markov interlacing property for the generalized polynomials is equivalent to the monotonicity of every zero η of the derivative $\phi'(\mathbf{x}; t)$ with respect to each of the zeros of $\phi(\mathbf{x}; t)$.

THEOREM 1. *Assume that $\bar{\varphi} := \{\varphi_1(x), ..., \varphi_n(x)\}$ is an arbitrary T-system of continuous functions on $[a, b]$ such that $\{\varphi_1(x), ..., \varphi_{n-1}(x)\}$ is also a T-system on $[a, b]$. Let $\bar{\phi} := \{\phi_0(x), ..., \phi_n(x)\}$ be the corresponding induced system. If one of the following two statements holds, then the other holds too*;

(i) *Assume that the points* $\mathbf{x} = (x_1, ..., x_n)$ *and* $\mathbf{y} = (y_1, ..., y_n)$ *interlace with at least one strict inequality* $x_i < y_i$. *Then the zeros of* $\phi'(\mathbf{x}; t)$ *and* $\phi'(\mathbf{y}; t)$ *interlace strictly.*

(ii) *Each zero* η *of* $\phi'(\mathbf{x}; t)$ *is a strictly increasing function of* x_k *on* $a \leqslant x_1 < \cdots < x_n \leqslant b$ *for each* $k \in \{1, ..., n\}$.

Proof. Assume that (ii) holds. Given $\mathbf{x}$ and $\mathbf{y}$, consider the family of polynomials $\phi(\mathbf{x}(\lambda); t)$, where

$$\mathbf{x}(\lambda) := (x_1(\lambda), ..., x_n(\lambda)) \qquad \text{and} \qquad x_k(\lambda) := (1-\lambda)\, x_k + \lambda y_k.$$

Clearly $\mathbf{x}(0) = \mathbf{x}$, $\mathbf{x}(1) = \mathbf{y}$. By the assumption, each zero $\eta_j(\lambda)$ $(j = 1, ..., n-1)$ of $\phi'(\mathbf{x}(\lambda); t)$ is a strictly increasing function of the zeros $x_1(\lambda), ..., x_n(\lambda)$. Then $\eta_j(\lambda)$ is a strictly increasing function of λ in $[0, 1]$. Thus $\eta_j(\lambda)$ will traverse, increasing monotonically, the interval $[t_j, \tau_j]$ defined by the zeros $\{t_i\}$ and $\{\tau_i\}$ of $\phi'(\mathbf{x}; t)$, $\phi(\mathbf{y}; t)$, respectively, when λ goes from 0 to 1. Hence

$$t_j \leqslant \eta_j(\lambda) \qquad \text{for} \quad \lambda \in [0, 1].$$

In order to prove that $t_1, ..., t_{n-1}$ interlace with $\eta_1(\lambda), ..., \eta_{n-1}(\lambda)$ for each $\lambda \in [0, 1]$, and particularly with $\tau_j := \eta_j(1)$, $j = 1, ..., n-1$, we have to show that

$$\eta_{j-1}(\lambda) \leqslant t_j \qquad \text{for} \quad \lambda \in [0, 1] \qquad \text{and} \qquad j = 2, ..., n-1.$$

Assume the contrary. Then $\eta_{j-1}(\lambda) = t_j$ for some $\lambda \in (0, 1]$ and therefore $\phi'(\mathbf{x}(\lambda); t_j) = \phi'(\mathbf{x}; t_j) = 0$. This conclusion contradicts Lemma 1 since $\mathbf{x}$ and $\mathbf{x}(\lambda)$ interlace. Therefore $\{\eta_j(\lambda)\}$ interlace with $\{t_i\}$ and the Markov property (i) is proved.

The converse follows easily: If (i) holds, then a small increase ε of x_k will produce two interlacing sequences

$$\mathbf{x} = (x_1, ..., x_n) \qquad \text{and} \qquad \mathbf{x}_\varepsilon := (x_1, ..., x_{k-1}, x_k + \varepsilon, x_{k+1}, ..., x_n)$$

with one strict inequality, namely $x_k < x_k + \varepsilon$. Then the corresponding sequences of the derivative zeros

$$t_1, ..., t_{n-1} \qquad \text{and} \qquad t_1(\varepsilon), ..., t_{n-1}(\varepsilon)$$

will interlace strictly. This implies $t_i < t_i(\varepsilon)$, which means strict monotonicity. The theorem is proved. ∎

Taking into account the remark concerning the monotonicity of η in case of algebraic polynomials, we can derive the Markov lemma from Theorem 1.

Remark. The main reason we gave Theorem 1 was to present the method. We did this for a certain class of T-systems. However the method applies to a more general situation involving multiple zeros, where some of the points $x_1, ..., x_n$ may coalesce. In this case one should consider Extended Tchebycheff systems $\psi_0, ..., \psi_n$ (i.e., such that any non-zero generalized polynomial $a_0\psi_0(t) + \cdots + a_n\psi_n(t)$ has no more than n zeros, counting multiplicities (see [8])). It is not difficult to see that a correspondingly modified version of Lemma 1, and consequently of Theorem 1, holds in this more general setting.

The trigonometric case is not governed by Theorem 1 but it can be proved in the same way even for any system of $(b-a)$-periodic functions $\psi_0, ..., \psi_{2n}$ which form a T-system on $[a, b)$ and such that the $\psi'_0, ..., \psi'_{2n}$ is also a T-system there. We omit the details.

The monotonicity in the trigonometric case can be shown using the identity

$$\frac{\tau'(x)}{\tau(x)} = \frac{1}{2} \sum_{k=1}^{2n} \cot \frac{x - x_k}{2}$$

which holds for every trigonometric polynomial

$$\tau(x) = \sin \frac{x - x_1}{2} \cdots \sin \frac{x - x_{2n}}{2}$$

with zeros $0 = x_1 < \cdots < x_{2n} < 2\pi$. Differentiating it with respect to x_k at the zero η of τ' we get

$$\frac{\partial \eta}{\partial x_k} = \frac{1}{\sin^2(\eta - x_k)/2} \Bigg/ \left(\sum_{i=1}^{2n} \frac{1}{\sin^2(\eta - x_i)/2} \right) > 0$$

and thus η is strictly monotone. This proves the Markov interlacing property for trigonometric polynomials.

Let u and v be two trigonometric polynomials of degree n with $2n$ distinct zeros $\{x_i\}_1^{2n}$ and $\{y_i\}_1^{2n}$, respectively, such that

$$x_1 \leqslant y_1 \leqslant \cdots \leqslant x_{2n} \leqslant y_{2n} < x_1 + 2\pi$$

with at least one strict inequality $x_i < y_i$. Then the zeros of u' and v' interlace strictly, that is,

$$t_1 < \tau_1 < \cdots < t_{2n} < \tau_{2n} < t_1 + 2\pi.$$

The Monotonicity. This is the difficult part of proving that certain systems possess the Markov interlacing property. In the algebraic and the trigonometric case this was proven using the explicit expression of the ratio f'/f in terms of the zeros of f. Unfortunately there is no such simple relation in the case of generalized polynomials. We present here another method of establishing monotonicity for a quite general class of T-systems.

THEOREM 2. *Let $\bar{\varphi}$ and $\bar{\phi}$ be given as in Theorem 1. Then each zero η of $\phi'(\mathbf{x}; t)$ is a strictly increasing function of x_k $(k = 1, ..., n)$ in $a \leqslant x_1 < \cdots < x_n \leqslant b$.*

Proof. Let $\mathbf{x} = (x_1, ..., x_n)$ be an arbitrary set of points such that $a \leqslant x_1 < \cdots < x_n \leqslant b$. For fixed k and sufficiently small $\varepsilon > 0$, consider the set

$$\mathbf{x}_\varepsilon := (x_1, ..., x_{k-1}, x_k + \varepsilon, x_{k+1}, ..., x_n).$$

Denote by $\{\eta_j(\varepsilon)\}_1^{n-1}$ the zeros of $\phi'(\mathbf{x}_\varepsilon; t)$. We claim that $\eta_j(\varepsilon) > \eta_j(0)$ for a sufficiently small $\varepsilon > 0$ and each $j = 1, ..., n-1$. This would imply the theorem.

Assume the contrary. Then $\eta_j(\varepsilon) \leqslant \eta_j(0)$ for some j and some small $\varepsilon > 0$. The equality can not happen because of Lemma 1. Thus the inequality above is strict. Let us now move x_j (respectively $x_k + \varepsilon$, in the case $j = k$) ahead, towards x_{j+1}, denoting the new position of x_j by x_j^+. Since all the time we have $x_j^+ < \eta_j^+ < x_{j+1}$, where η_j^+ is the critical point associated with the new system of points, and since $\eta_j(\varepsilon) < \eta_j^+$ for $x_j^+ = \eta_j(\varepsilon)$, we will get $\eta_j(\varepsilon) = \eta_j^+$ for some x_j^+ between x_j and $\eta_j(\varepsilon)$. But this contradicts Lemma 1, since $\mathbf{x}$ and the translated system (with x_j^+ and $x_k + \varepsilon$ instead of x_j and x_k, in case $j \neq k$, and with x_j^+ instead of x_k, in case $j = k$) interlace. Thus, we proved that η_j is a strictly increasing function of x_k. The theorem is proved. ∎

As an immediate consequence of Theorem 1 and Theorem 2 we get the following extension of the Markov lemma.

COROLLARY 1. *Let $\bar{\phi} = \{1, \phi_1(t), ..., \phi_n(t)\}$ be an arbitrary system of continuously differentiable functions on $[a, b]$ (with $\phi_0(t) = 1$) such that $\phi'_1, ..., \phi'_k$ constitute a T-system for $k = n-1$ and $k = n$. Suppose that the zeros of $\phi(\mathbf{x}; t)$ and $\phi(\mathbf{y}; t)$ interlace with at least one strict inequality. Then the zeros of $\phi'(\mathbf{x}; t)$ and $\phi'(\mathbf{y}; t)$ interlace strictly.*

3. PERFECT SPLINES

We shall assume in this section that r and n are fixed integers such that $r > 0$, $n \geqslant r$. As we mentioned in the introduction, for any system of points

$\mathbf{x}=(x_1, ..., x_n)$, $x_1 < \cdots < x_n$, there exists a unique perfect spline $p(\mathbf{x}; t)$ of degree r with at most $n-r$ knots which satisfies the conditions

$$p(\mathbf{x}; x_k)=0, \qquad k=1, ..., n,$$

and is normalized by $p^{(r)}(\mathbf{x}; x_n)=1$. We are going to study the Markov interlacing property for $p(\mathbf{x}; t)$. First we recall some known facts concerning $p(\mathbf{x}; t)$ which will be needed in the sequel (see [2] for details).

(a) For each $x_1 < \cdots < x_n$ the spline $p(\mathbf{x}; t)$ has no other zeros except $x_1, ..., x_n$ and all they are simple.

(b) $p(\mathbf{x}; t)$ has exactly $n-r$ knots. We shall denote them by $\xi_1, ..., \xi_{n-r}$.

(c) The zeros $\mathbf{x}$ and the knots ξ of the perfect spline $p(\mathbf{x}; t)$ satisfy the Schoenberg–Whitney interlacing condition

$$\xi_{i-r} < x_i < \xi_i, \qquad i=1, ..., n, \tag{2}$$

when meaningful.

(d) $p'(\mathbf{x}; t)$ has exactly $n-1$ zeros $\eta_1 < \cdots < \eta_{n-1}$ and all they are simple. (In case $r=1$ by a "zero" of $p'(\mathbf{x}; t)$ we mean a "sign change." Moreover,

$$x_1 < \eta_1 < x_2 < \cdots < \eta_{n-1} < x_n$$

and $\{\eta_j\}_1^{n-1}$ are extremal points for $p(\mathbf{x}; t)$.

We separate the next property as a lemma.

LEMMA 2. *Let $\{\eta_j\}_1^{n-1}$ be the zeros of $p'(\mathbf{x}; t)$. For each $r>1$ we have $p''(\mathbf{x}; \eta_j) \neq 0$ and*

$$\operatorname{sign} p''(\mathbf{x}; \eta_j)=(-1)^{n-j-1}, \qquad j=1, ..., n-1.$$

Proof. Since $p'(\mathbf{x}; t)$ is a perfect spline with $n-r$ knots vanishing at $n-1$ distinct points, it has by virtue of property (a) only simple zeros. Thus $p''(\mathbf{x}; \eta_j) \neq 0$ and the quantities $p''(\mathbf{x}; \eta_1), ..., p''(\mathbf{x}; \eta_{n-r})$ are of alternating sign. The sign can be determined from the normalization $p^{(r)}(\mathbf{x}; x_n)=1$ which implies $p''(\mathbf{x}; t)>0$ for $t>\xi_{n-r}$, and as a consequence, $p''(\mathbf{x}; \eta_{n-1})>0$. The lemma is proved. ∎

In the simplest case when $r=1$ the perfect spline $p(\mathbf{x}; t)$ is a piece-wise linear function which vanishes at $x_1, ..., x_n$ and the extremal points $\{\eta_j\}_1^{n-1}$ are given by $\eta_j=(x_j+x_{j+1})/2=\xi_j$, $j=1, ..., n-1$. Clearly the increase ε of x_k will cause an increase by $\varepsilon/2$ of η_{k-1} and η_k only, while the other extremal points η_j will not change at all. Thus, we cannot expect that every η_j is a strictly increasing function of any x_k as it was in the smooth case of generalized polynomials (compare with the results in the previous section). On the other hand, if (again in the case $r=1$) all zeros $x_1, ..., x_n$ increase, then every extremal point η_j will also increase. We shall show that such a monotonicity takes place for perfect splines of any degree r and this will imply Markov's interlacing property in a somewhat weaker form for splines.

First we derive certain conclusions concerning Markov's interlacing property for perfect splines following the method described in the previous section. Then we go to a more careful study of the behaviour of η as a function of x_k using the total positivity structure of the spline kernel.

We start with the following observation which will be referred to as the "spline version of Lemma 1."

Assume that $\mathbf{x} = (x_1, ... x_n)$ *and* $\mathbf{y} := (y_1, ..., y_n)$ *interlace strictly, that is,*

$$x_1 < y_1 < x_2 < \cdots < x_n < y_n. \tag{3}$$

Then there is no point η for which $p'(\mathbf{x}; \eta) = p'(\mathbf{y}; \eta) = 0$.

The case $r=1$ was proven above. We prove this for $r>1$ following the proof of Lemma 1. Let us point out first the fact that in case $r=2$ no extremal point η of the perfect $p(\mathbf{x}; \eta)$ could coincide with a knot ξ of $p(\mathbf{x}; \eta)$. Indeed, otherwise the perfect spline $p'(\mathbf{x}; t)$ (which is of degree 1 and has exactly $n-1$ zeros) would have $n-1$ knots: $(\eta_i + \eta_{i+1})/2$, $i = 1, ..., n-2$, and that particular $\eta = \xi$. This would contradict property (b).

Assume now that the assertion does not hold. Then there is a common zero η of $p'(\mathbf{x}; t)$ and $p'(\mathbf{y}; t)$. Consider the spline

$$Q(t) := p(\mathbf{x}; t) - \alpha p(\mathbf{y}; t) \qquad \text{with} \quad \alpha = \frac{p(\mathbf{x}; \eta)}{p(\mathbf{y}; \eta)}.$$

We have $Q(\eta) = Q'(\eta) = 0$. Because of (3) any interval (x_i, x_{i+1}), $i = 1, ..., n-1$, contains an odd number of zeros of Q (since Q changes sign on (x_i, x_{i+1})). Therefore Q has at least 1 zero in (x_i, x_{i+1}) for $i \neq j$ and at least 3 zeros, counting the multiplicities, in the interval (x_j, x_{j+1}) containing η. Note that the zeros in (x_i, x_{i+1}) are isolated from those in the neighboring subintervals. Then by Rolle's theorem Q' will have at least $n-2$ sign changes and a double zero (if all above mentioned 3 zeros in (x_j, x_{j+1}) coincide with η) or at least n sign changes (if η does not coincide with the other third zero of Q in the subinterval (x_j, x_{j+1})). Applying again Rolle's theorem we conclude that Q'' will have at least $n-1$ sign changes and finally $Q^{(r)}(t)$ will have at least $n-r+1$ sign changes. But if $|\alpha| > 1$, then sign $Q^{(r)}(t) = -\text{sign } p^{(r)}(\mathbf{y}; t)$ and we arrive at contradiction since

$p(\mathbf{y}; t)$ has exactly $n-r$ knots. Similarly we get a contradiction in case $|\alpha| \leqslant 1$ since then sign $Q^{(r)}(t) = \text{sign } p^{(r)}(\mathbf{x}; t)$. Our claim is proved. ▮

Remark. In case $\mathbf{x}$ and $\mathbf{y}$ do not interlace strictly (i.e., not all inequalities $x_i \leqslant y_i$ are strict), we can not apply the reasoning as in the polynomial case and show that Q'' has at least $n-1$ sign changes, since the spline Q (unlike the polynomial) could vanish on a subinterval covering many of the zeros $\{x_i\}$.

Assume now that η is any zero of $p'(\mathbf{x}; t)$ of a fixed index. Suppose that $\mathbf{x}^0 = (x_1^0, ..., x_n^0)$, $x_1^0 < \cdots < x_n^0$, is a fixed set of points in $[a, b]$. Consider η at $\mathbf{x} = (x_1, ..., x_n)$ with $x_i = x_i^0$ for $i \neq k$ and x_k free in a neighborhood $\mathscr{U}$ of x_k^0. Then

$$p'(\mathbf{x}; \eta(\mathbf{x})) \equiv 0$$

for each $x_k \in \mathscr{U}$. The function $\eta(\mathbf{x})$ is defined implicitly in $\mathscr{U}$ by the equation $F(\eta) := p'(\mathbf{x}; \eta) = 0$. Note that

$$\frac{\partial}{\partial t} p'(\mathbf{x}; t)|_{t=\eta} = p''(\mathbf{x}; \eta)$$

and hence, by virtue of Lemma 2, $F'(\eta) \neq 0$. Then, by the Implicit Function Theorem, η is differentiable in $\mathscr{U}$. In Theorem 4 we give an explicit expression for $\partial\eta/\partial x_k$.

LEMMA 3. *Let η be a zero of $p'(\mathbf{x}; t)$. Then*

$$\frac{\partial \eta}{\partial x_k} \geqslant 0.$$

Proof. For a fixed k, consider the set $\mathbf{x}_\varepsilon$ obtained from $\mathbf{x}$ by an ε increase of x_k. Assume that the corresponding zero $\eta(\varepsilon)$ of $p(\mathbf{x}(\varepsilon); t)$ is smaller than $\eta(0) = \eta$. Set $\delta := \eta - \eta(\varepsilon)$. Then we get two splines $p(\mathbf{x}; t)$ and $p(\mathbf{y}; t) := p(\mathbf{x}_\varepsilon; t - \delta)$ with strictly interlacing zeros and then the spline version of Lemma 1 leads to contradiction. Thus $\eta(\varepsilon) \geqslant \eta(0)$ for each sufficiently small ε. Since η is a differentiable function of x_k (and consequently of ε), we get $\eta'(0) \geqslant 0$. The proof is completed. ▮

THEOREM 3. *Let $\mathbf{x}$ and $\mathbf{y}$ interlace. Then the zeros of $p'(\mathbf{x}; t)$ and $p'(\mathbf{y}; t)$ also interlace. If the points $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ satisfy the strict interlacing conditions*

$$x_1 < y_1 < x_2 < \cdots < x_n < y_n,$$

then the zeros of $p'(\mathbf{x}; t)$ and $p'(\mathbf{y}; t)$ strictly interlace.

Proof. Having Lemma 3, we prove that the zeros $\eta(\lambda)$ of $p'(\mathbf{x}(\lambda); t)$ (where $x_i(\lambda) := (1-\lambda) x_i + \lambda y_i$) increase (not necessarily strictly) when λ goes from 0 to 1, remaining all time between the corresponding zeros of $p'(\mathbf{x}; t)$ and $p'(\mathbf{y}; t)$. If, in addition, $x_i < y_i$ for all i, then by virtue of the spline version of Lemma 1 every zero η_j of $p'(\mathbf{x}; t)$ is distinct from the corresponding zero of $p'(\mathbf{y}; t)$ and therefore the zeros of the derivatives interlace strictly. We omit the details since they are given in the proof of Theorem 2. ▮

Note that we can not conclude from Theorem 3 that the interlacing of $\mathbf{x}$ and $\mathbf{y}$ with at least one strict inequality implies strict interlacing of the zeros of $p'(\mathbf{x}; t)$ and $p'(\mathbf{y}; t)$. In order to get a proposition of this kind we have to study carefully the dependence of η on the zeros $x_1, ..., x_n$. This is our next task.

Denote by $S_{r-1}(\xi_1, ..., \xi_{n-r})$ the linear space of all spline functions of degree $r-1$ with knots $\xi_1, ..., \xi_{n-r}$. According to property (c) of the perfect splines, the zeros $\mathbf{x} = (x_1, ..., x_n)$ and the knots $\xi = (\xi_1, ..., \xi_{n-r})$ of $p(\mathbf{x}; t)$ satisfy the interlacing condition (2). Then, by a well-known result due to Schoenberg an Whitney [16], the collocation matrix

$$J := \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{r-1} & (x_1-\xi_1)_+^{r-1} & \cdots & (x_1-\xi_{n-r})_+^{r-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{r-1} & (x_n-\xi_1)_+^{r-1} & \cdots & (x_n-\xi_{n-r})_+^{r-1} \end{bmatrix}$$

is non-singular. Thus the interpolation problem

$$s(x_k) = f_k, \qquad k = 1, ..., n,$$

by splines $s \in S_{r-1}(\xi_1, ..., \xi_{n-r})$ has a unique solution for each given data $\{f_k\}$. Denote by $\{l_k(\mathbf{x}; t)\}_{k=1}^{n}$ the Lagrange fundamental functions for the above interpolation problem. In other words, $l_k(\mathbf{x}; t)$ is the unique spline from $S_{r-1}(\xi_1, ..., \xi_{n-r})$ satisfying the conditions

$$l_k(\mathbf{x}; x_i) = \delta_{ki}, \qquad i = 1, ..., n$$

(δ_{ik} being the Kronecker symbol). Then every spline $f \in S_{r-1}(\xi_1, ..., \xi_{n-r})$ can be presented in the form

$$f(t) = \sum_{k=1}^{n} f(x_k)\, l_k(\mathbf{x}; t). \tag{4}$$

The following property of $l_k(\mathbf{x}; t)$ is crucial in our study of $\partial\eta/\partial x_k$.

LEMMA 4. *If the zeros* $\mathbf{x}$ *and the knots* ξ *of the perfect spline* $p(\mathbf{x}; t)$ *satisfy the stronger Schoenberg–Whitney interlacing conditions*

$$\xi_{i-r+1} < x_i < \xi_{i-1}, \tag{5}$$

then $l_k(\mathbf{x}; t) \neq 0$ *for each* $t \notin \{x_1, ..., x_n\}$ $(k = 1, ..., n)$. *Moreover, for each zero* η *of* $p'(\mathbf{x}; t)$, *we have*

$$\sigma(-1)^k \, l'_k(\mathbf{x}; \eta) > 0$$

with $\sigma = 1$ *or* $\sigma = -1$, *depending only on* η.

Proof. Note that the conditions (5) cannot hold for $r = 1$ and $r = 2$. We assume below that $r > 2$.

The Lagrange fundamental spline $l_k(\mathbf{x}; t)$ has the determinantal representation

$$l_k(\mathbf{x}; t) = \frac{\det J_k(t)}{\det J},$$

where

$$J_k(t) := \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{r-1} & (x_1 - \xi_1)_+^{r-1} & \cdots & (x_1 - \xi_{n-r})_+^{r-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & t & \cdots & t^{r-1} & (t - \xi_1)_+^{r-1} & \cdots & (t - \xi_{n-r})_+^{r-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{r-1} & (x_n - \xi_1)_+^{r-1} & \cdots & (x_n - \xi_{n-r})_+^{r-1} \end{bmatrix}$$

(the row involving t is the kth row). Indeed, the numerator $\det J_k(t)$ is a spline from $S_{r-1}(\xi_1, ..., \xi_{n-r})$ and $\det J_k(x_i) = 0$ for $i \neq k$, $\det J_k(x_k) = \det J$. Thus $\det J_k(t)/\det J$ satisfies the interpolation conditions which determine $l_k(\mathbf{x}; t)$ uniquely. Observe now that the points

$$(t_1, ..., t_n) = (x_1, ..., x_{k-1}, t, x_{k+1}, ..., x_n)$$

in increasing order and the knots ξ satisfy the interlacing conditions

$$\xi_{i-r} < x_{i-1} \leqslant t_i \leqslant x_{i+1} < \xi_i$$

for each $t \in (-\infty, \infty)$. Thus $\det J_k(t) \neq 0$ and consequently $l_k(\mathbf{x}; t) \neq 0$ for each $t \neq x_1, ..., x_n$. It also follows from the determinant expression of $J_k(t)$ (by considering the order of the rows of the matrix $J_k(t)$) that $l_k(\mathbf{x}; t)$ changes sign when t passes through a zero x_i, $i \neq k$.

It remains to show that $l'_k(\mathbf{x}; \eta)$ changes sign alternatively when k varies from 1 to n. Assume the contrary. Then there is a k such that

$$l'_k(\mathbf{x}; \eta)\, l'_{k+1}(\mathbf{x}; \eta) > 0.$$

Define the spline

$$q_\lambda(t) := (1-\lambda)\, l_k(\mathbf{x}; t) - \lambda l_{k+1}(\mathbf{x}; t).$$

Clearly $q_0(t) = l_k(\mathbf{x}; t)$ while $q_1(t) = -l_{k+1}(\mathbf{x}; t)$. The lemma will be proved if we show that $q'_\lambda(\eta)$ does not vanish for $\lambda \in (0, 1)$. Assume that $q'_\lambda(\eta) = 0$ for some $\lambda \in (0, 1)$. Let $\eta \in (x_j, x_{j+1})$. Suppose first that $j \neq k$. Then we consider the spline $s(t) := p(\mathbf{x}; t) - \alpha q_\lambda(t)$ with $\alpha = p(\mathbf{x}; \eta)/q_\lambda(\eta)$. Note that $q_\lambda(\eta) \neq 0$ since $l_k(\mathbf{x}; t)$ and $-l_{k+1}(\mathbf{x}; t)$ have the same sign on (x_j, x_{j+1}).

Clearly $s(\eta) = 0$ by construction and $s'(\eta) = p'(\mathbf{x}; \eta) - \alpha q'_\lambda(\eta) = 0$, by the assumption. Thus s has a double zero at η. Note now that $s(t)$ vanishes also at certain point $\tau \in (x_k, x_{k+1})$ since

$$s(x_k) = -\alpha q_\lambda(x_k) = -\alpha(1-\lambda)$$

$$s(x_{k+1}) = \alpha\lambda$$

and thus $s(x_k)\, s(x_{k+1}) < 0$. Adding the obvious zeros $x_1, ..., x_{k-1}, x_{k+2}, ..., x_n$ of s we see that s has at least $n+1$ zeros in $(-\infty, \infty)$. All are isolated since $p(\mathbf{x}; t)$, and hence $s(t)$, is a polynomial of degree r with a non-zero leading coefficient on each subinterval (ξ_i, ξ_{i+1}), $i = 0, ..., n-r$, ($\xi_0 := -\infty$, $\xi_{n-r+1} := \infty$). By Rolle's theorem, $s^{(r-1)}(t)$ should have at least $n-r+2$ sign changes in $(-\infty, \infty)$. But $s^{(r-1)}(t)$ is monotone on each subinterval (ξ_i, ξ_{i+1}) changing alternatively the type of monotonicity (increasing, decreasing) in each of the subsequent subintervals. This observation follows from the fact that

$$s^{(r)}(t) = p^{(r)}(\mathbf{x}; t) \qquad \text{on} \quad (\xi_i, \xi_{i+1})$$

and $p^{(r)}(\mathbf{x}; t)$ changes sign alternatively. Now it is seen that such a piece wise monotone function can have at most one sign change in $[\xi_i - 0, \xi_{i+1})$, $i = 0, ..., n-r$, and thus at most $n-r+1$ sign changes in $(-\infty, \infty)$. We arrive at contradiction in case $\eta \notin (x_k, x_{k+1})$.

Assume now that $\eta \in (x_k, x_{k+1})$. If $q_\lambda(\eta) \neq 0$, then we can proceed as in the previous case and construct the spline $s(t)$. By construction $s(t)$ vanishes at $x_1, ..., x_{k-1}, x_{k+2}, ..., x_n$ and has a double zero at η. Since $s(x_k)\, s(x_{k+1}) < 0$ and $\eta \in (x_k, x_{k+1})$, we conclude that s has at least 3 zeros in (x_k, x_{k+1}) counting multiplicities, and hence totally $n+1$ zeros in $(-\infty, \infty)$. As we saw already Rolle's theorem then leads to a contradiction. Thus the proof of the lemma will be completed if we show that $q_\lambda(\eta) \neq 0$.

In order to do this, recall first that $q_\lambda(t)$ vanishes at $x_1, ..., x_{k-1}, x_{k+2}, ..., x_n$ and these are isolated zeros. Assume in addition that $q_\lambda(\eta)=0$. Then η would be at least a double zero of $q_\lambda(t)$. Since $q_\lambda(x_k)\, q_\lambda(x_{k+1})<0$, we conclude that $q_\lambda(t)$ has at least three zeros in (x_k, x_{k+1}), counting the multiplicities (remember that r was assumed greater than 2). Then Rolle's theorem will imply that $q_\lambda^{(r-1)}(t)$ has at least $n-r+2$ sign changes. But this is impossible since $q_\lambda(t)$ is a spline of degree $r-1$ with $n-r$ knots, namely $\xi_1, ..., \xi_{n-r}$. Thus $q_\lambda(\eta)\neq 0$ and the lemma is proved. ∎

THEOREM 4. *Assume the zeros* $\mathbf{x}$ *and the knots* ξ *of the perfect spline* $p(\mathbf{x}; t)$ *satisfy the stronger Schoenberg–Whitney interlacing conditions* (5). *Assume that the points* $\mathbf{x}=(x_1, ..., x_n)$ *and* $\mathbf{y}=(y_1, ..., y_n)$ *interlace with at least one strict inequality* $x_i<y_i$. *Then the zeros of* $p'(\mathbf{x}; t)$ *and* $p'(\mathbf{y}; t)$ *interlace strictly.*

Proof. Let η be a zero of $p'(\mathbf{x}; t)$. The theorem will be proved if we show that $\partial\eta/\partial x_k>0$ for each k. To this aim, we next find an appropriate expression for the derivative.

As we mentioned already (see Theorem A), given $\mathbf{x}$, the coefficients $\alpha_1, ..., \alpha_n$ and the knots $\xi_1<\cdots<\xi_{n-r}$ of the perfect spline

$$p(\mathbf{x}; t)=\sum_{j=1}^{r} \alpha_j t^{j-1}+c\left[t^r+2\sum_{i=1}^{n-r}(-1)^i\,(t-\xi_i)_+^r\right]$$

(with $c=(-1)^{n-r}/r!$) are uniquely determined as solutions of the nonlinear system of equations

$$\sum_{j=1}^{r} \alpha_j x_k^{j-1}+c\left[x_k^r+2\sum_{i=1}^{n-r}(-1)^i\,(x_k-\xi_i)_+^r\right]=0$$

for $k=1, ..., n$. Denote by $D=D(x_1, ..., x_n)$ the Jacobian determinant of this system with respect to $\alpha_1, ..., \alpha_r, \xi_1, ...\xi_{n-r}$. Evidently

$$D=C\det J$$

with some non-zero constant C, and therefore $D\neq 0$ for each $\mathbf{x}$ with $x_1<\cdots<x_n$. By the Implicit Function Theorem, the $\{\alpha_j\}$ and $\{\xi_i\}$ are differentiable functions of $x_1, ..., x_n$. Moreover,

$$\frac{\partial\alpha_j(\mathbf{x})}{\partial x_k}=-\frac{A_{kj}}{D}, \qquad j=1, ..., r,$$

$$\frac{\partial\xi_i(\mathbf{x})}{\partial x_k}=-\frac{A_{k,r+i}}{D}, \qquad i=1, ..., n-r,$$

where A_{kj} is obtained from D replacing column j by the column

$$(0, ..., 0, p'(\mathbf{x}; x_k), 0, ..., 0)^T$$

with the non-zero entry in the kth position. Then clearly

$$A_{kj} = p'(\mathbf{x}; x_k)(-1)^{k+j} D_{kj},$$

where D_{kj} is the corresponding subdeterminant of D obtained by deleting row k and column j.

To find $\partial\eta/\partial x_k$ we shall use the identity

$$\frac{\partial}{\partial x_k} p'(\mathbf{x}; \eta(\mathbf{x})) \equiv 0.$$

Since

$$p'(\mathbf{x}; \eta) = \sum_{j=2}^{r} (j-1)\, \alpha_j \eta^{j-2} + rc \left[\eta^{r-1} + 2 \sum_{i=1}^{n-r} (-1)^i\, (\eta - \xi_i)_+^{r-1} \right],$$

performing the differentiation with respect to x_k, we get

$$\sum_{j=2}^{r} (j-1)\, \eta^{j-2} \frac{\partial \alpha_j}{\partial x_k}$$
$$+ 2r(r-1)\, c \sum_{i=1}^{n-r} (-1)^{i+1}\, (\eta - \xi_i)_+^{r-2} \frac{\partial \xi_i}{\partial x_k} + p''(\mathbf{x}; \eta) \frac{\partial \eta}{\partial x_k} = 0.$$

Now making use of the explicit expression for the derivative of α_j and ξ_i we rewrite the last equality in the form

$$-\frac{p'(\mathbf{x}; x_k)}{D} \left[\sum_{j=2}^{r} (j-1)\, \eta^{j-2} (-1)^{k+j} D_{kj} + 2r(r-1)\, c \right.$$
$$\left. \times \sum_{i=1}^{n-r} (-1)^{i+1}\, (\eta - \xi_i)_+^{r-2}\, (-1)^{k+r+i} D_{k,\, r+i} \right] = p''(\mathbf{x}; \eta) \frac{\partial \eta}{\partial x_k}.$$

Observe that the expression in the square brackets above is just the expansion of the determinant

$$B_k := \det \begin{bmatrix} \psi_0(x_1) & \psi_1(x_1) & \cdots & \psi_{n-1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \psi_1'(\eta) & \cdots & \psi_{n-1}'(\eta) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(x_n) & \psi_1(x_n) & \cdots & \psi_{n-1}(x_n) \end{bmatrix}$$

(with $\psi_j(t) := t^{j-1}$, $j = 1, ..., r$, $\psi_{r+i} := 2cr(-1)^i (t - \xi_i)_+^{r-1}$, $i = 1, ..., n-r$) along its kth row (which is written explicitly in the determinant above). Clearly

$$B_k = \frac{\partial}{\partial x} D(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_n)|_{x=\eta}.$$

But $D(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_n)/D$ is the spline s from $S_{r-1}(\xi_1, ..., \xi_{n-r})$ satisfying the conditions

$$s(x_i) = \delta_{ik}, \qquad i = 1, ..., n.$$

Thus $s = l_k(\mathbf{x};\ \cdot\ )$. Taking into account all these facts we finally obtain

$$-p'(\mathbf{x}; x_k)\, l_k'(\mathbf{x}; \eta) + p''(\mathbf{x}; \eta) \frac{\partial \eta}{\partial x_k} = 0.$$

Therefore

$$\frac{\partial \eta}{\partial x_k} = \frac{p'(\mathbf{x}; x_k)}{p''(\mathbf{x}; \eta)}\, l_k'(\mathbf{x}; \eta). \tag{6}$$

By Lemma 4, $l_k'(\mathbf{x}; \eta) \neq 0$. Thus $\partial\eta/\partial x_k \neq 0$. Then Lemma 3 yields $\partial\eta/\partial x_k > 0$. The theorem is proved. ▮

Remark. Having (6) we can complete the proof of the theorem without recourse to Lemma 3. We give below this alternative proof since the idea could be used in the study of other systems for which the proof of Lemma 3 would not work (it uses the assumption that the translation of any generalized polynomial is also a generalized polynomial).

Since $p'(\mathbf{x}; t) \in S_{r-1}(\xi_1, ..., \xi_{n-r})$, it follows from (4) that

$$p'(\mathbf{x}; t) = \sum_{k=1}^{n} p'(\mathbf{x}; x_k)\, l_k(\mathbf{x}; t)$$

and therefore

$$p''(\mathbf{x}; \eta) = \sum_{k=1}^{n} p'(\mathbf{x}; x_k)\, l_k'(\mathbf{x}; \eta).$$

But both factors $p'(\mathbf{x}; x_k)$ and $l_k'(\mathbf{x}; \eta)$ alternate in sign as k goes from 1 to n. Thus every term in the sum above is of the same sign σ and clearly $\sigma = \operatorname{sign} p''(\mathbf{x}; \eta)$. Now (6) implies $\partial\eta/\partial x_k > 0$.

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